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A New Transform for Time-Frequency Analysis

Arun Kumar, Daniel R. Fuhrmann, Michael Frazier, and Bjorn Jawerth

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A NEW TRANSFORM FOR TIME-FREQUENCY ANALYSIS

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ABSTRACT

This paper describes how a signal can be written as a weighted sum of certain "elementary" synthesizing functions, which are the dilated and translated versions of a single parent function. The weighting constants in this sum define a transform of the signal. This is much like Fourier analysis except that a wide choice is permitted in the selection of a set of synthesizing functions. Moreover, the permitted sets of snythesizing functions are not orthogonal. It is shown that the transform described here captures both the frequency content, and the temporal evolution, of a non-stationary signal.

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1. Introduction

If a signal consists of two pure tones played in succession, we cannot tell where the signal changed character by looking at a plot of its Fourier transform. Fourier methods yield a pure frequency description of a signal. For a non–stationary signal, one whose "frequency"–content changes with time, we would like a time–frequency description [1], [2]. Speech spectrograms are an example of a time–frequency description obtained by computing the Fourier transforms of a signal as seen through a window that slides along the length of the signal. The problem with such an approach is that there is no way to determine what the width of the window should be. If the window is too wide, we can fail to capture the non–staionarities in a signal. If too narrow, we lose the ability to resolve low–frequencies.

The Wigner distribution [3] was invented in 1932 in order to investigate the space–momentum localization of quantum–mechanical particles. Since space–momentum uncertainity and time–frequency uncertainity are analogous phenomena [2], the Wigner distribution has been developed as a tool for time–frequency analysis [4]–[6]. However, the Wigner distribution does not give us an accurate time–frequency description. For example, if a signal is the sum of two pure sinusoids at frequencies ω_1 and ω_2, then its Wigner distribution will incorrectly show the presence of the beat frequencies $(\pm\omega_1 \pm \omega_2)$ [6].

In this paper we will describe the Phi–transform [7]–[9], which was developed in 1984 in order to study the smoothness, size, and cancellation properties of functions, but can also be used to obtain the time–frequency description of a signal. The essential idea behind the Phi–transform is to generate a family of analyzing functions that are compactly–supported (or concentrated) in either the time or the frequency domain, and that are small and rapidly–decaying outside a compact–support in the complementary domain. We can "probe" a given signal with each of these analyzing functions by computing a sequence of inner–products. The inner–product of the signal with an analyzing function yields information about the signal in those regions of the time–frequency space where the analyzing function lives. In contrast, Fourier analysis employs analyzing functions that extend indefinitely in the time–domain, while they are perfectly concentrated in the frequency–domain.

In classical engineering literature (see for example Shannon [10], p. 13) signals are sometimes represented as sums of functions like $(\sin t)/t$. The Fourier transform of such a function is a rectangular pulse, while $(\sin t)/t$ itself decays only as t^{-1}. The Phi–transform is based upon a similar expansion (or decomposition) of a function, except that here we use functions whose Fourier transforms are smooth, not rectangular. The result is analyzing functions that decay rapidly in time, and yield simultaneous localization in time and frequency.

In 1985, Lemarié and Meyer [11] introduced the Wavelet transform. The Wavelet transform also gives a decomposition of a signal as a weighted sum of functions which are, as in the case of the Phi–transform, concentrated in both the time and the frequency domains. The main advantage of the Wavelet transform is that the analyzing functions, called *wavelets*, form an orthonormal set. Although orthogonality may be desirable in some circumstances, it is not essential in many others [12]. The advantage of the Phi–transform is that it is simpler and less rigid; in particular, greater flexibility is permitted in the choice of the analyzing functions. The time–frequency representations obtained by the Wavelet transformation of signals have been studied extensively [13]–[19].

We propose the Phi–transform as a tool of wide and general applicability in signal processing, that will be of use whenever we encounter non–stationary behaviour. We consider the Phi–transform to be a method for time–frequency analysis that is superior to the classical methods of Gabor and Wigner. In comparison to the Wavelet transform, we consider the Phi–transform to be simpler to understand, and apply.

2. Notation

Throughout, by the constant n we mean the dimension of the space $\mathbf{R}^n$ in which we work. $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$, and C^∞, denote the set of all integers (positive, negative, and zero), the set of real numbers, the set of complex numbers, and the set of infinitely–differentiable functions, respectively. By $\mathbf{R}^+$ we mean the set of positive real numbers. By L^2, and l^2, we denote the Hilbert spaces of absolutely square–integrable functions, and of absolutely square–summable sequences, respectively.

If $a, b \in \mathbf{R}^n$ are vectors, then by $a \cdot b$ we mean their scalar (or dot) product. The Euclidean norm of a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ will be written $\|\omega\| = (\sum_{i=1}^n |\omega_i|^2)^{1/2}$. For $c, d \in \mathbf{R}$, $[c,d]$ is the closed interval from c to d, and $[c,d]^n \triangleq \prod_{i=1}^n [c,d]$. Also,

$$\int_{[c,d]^n} f(x)\,dx \triangleq \int_c^d \cdots \int_c^d f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n. \tag{1}$$

By the *support* of a function $f : \mathbf{R}^n \to \mathbf{C}$ we will mean the topological closure of the set of those points $x \in \mathbf{R}^n$ for which $f(x) \neq 0$. We will write *supp* f for this set. We say that *supp* f is *compact* if, for some $r > 0$, *supp* $f \subseteq \{x \in \mathbf{R}^n : \|x\| \leq r\}$. A bar drawn above a complex constant, or a complex–valued function, will denote the complex–conjugate of the constant, or the function. By $\tilde{f}\,(t)$ we mean $\overline{f(-t)}$. If f and g are functions from $\mathbf{R}^n$ to $\mathbf{C}$, then by $\langle f, g\rangle$ we mean the inner–product of f and g:

$$\langle f, g\rangle \triangleq \int_{\mathbf{R}^n} f(x)\,\overline{g(x)}\,dx = \overline{\langle g, f\rangle}. \tag{2}$$

By "$\hat{\ }$" we denote the forward, by "$^\vee$" the inverse, Fourier transform. We will use a few results from the Fourier series and Fourier transform representations of a function. These are listed in the appendix.

In what follows we will be working with sets of functions that are all obtained from a single parent function through a process of dilation and translation. We now establish a notation for such functions. We define $\hat{\phi}_\nu(\omega) \triangleq \hat{\phi}(2^{-\nu}\omega)$, $\nu \in \mathbf{R}$, $\omega \in \mathbf{R}^n$, as a dilation of ϕ in the frequency–domain. We define $\phi_\nu(t) \triangleq 2^{n\nu}\phi(2^\nu t)$, $\nu \in \mathbf{R}$, $t \in \mathbf{R}^n$, as a dilation of ϕ in the time–domain. Then $(\phi_\nu)\hat{\ } = \hat{\phi}_\nu$, and $(\hat{\phi}_\nu)^\vee = \phi_\nu$. We also define $\phi_{\nu k}(t) \triangleq 2^{n\nu/2}\phi(2^\nu t - k)$ as a dilation–and–translation of ϕ in the time domain. Please note: $\phi_{\nu 0}(t) \neq \phi_\nu(t)$. This subscript notation for ϕ will also be used for the functions Φ, ψ, Ψ, and θ.

The transform and "decomposition" we discuss here apply to signals that belong to $\mathcal{S}'$, the space of "tempered distributions" (see e.g. [20]). $\mathcal{S}'$ is the dual space of $\mathcal{S}$, the Schwartz space of "smooth and rapidly–decreasing functions". A function f is *rapidly–decreasing* if f and all its partial derivatives decay at infinity at a rate faster than any polynomial.

$\mathcal{S}'$ is a very large space that properly includes L^2, and distributions such as the Dirac–delta and its derivatives. Here, in the interest of simplicity, we will state our results only for $f \in L^2$. We will not consider issues relating to convergence; except to say that when using equations with infinite sums, such as $f(t) = \sum_{i=0}^{\infty} g_i(t)$, we will here implicitly mean the equality in the special sense of L^2–convergence. That is, if $f_N \triangleq \sum_{i=0}^{N} g_i$, then $f = \sum_{i=0}^{\infty} g_i$ if and only if

$$lim_{N\to\infty} \|f - f_N\|_{L^2}^2 = lim_{N\to\infty} \int_{\mathbf{R}^n} |f - f_N|^2 \; dt = 0. \tag{3}$$

For a detailed study of convergence in $\mathcal{S}'$ and other spaces, of the decomposition discussed in this paper, see [7]–[9].

3. The Psi–decomposition of a Function

Let $\mathbf{S} \triangleq \{\Psi_{mk}, \psi_{\nu k}\}_{\nu,k}$; $m, \nu \in \mathrm{Z}$; m fixed; $\nu > m$; $k \in \mathrm{Z}^n$; be called a set of *synthesizing functions.* The functions Ψ_{mk} in $\mathbf{S}$ are the translates (in $\mathbf{R}^n$) of the single function Ψ_{m0}; and the $\psi_{\nu k}$ are all translated–and–dilated versions of a single function ψ. Similarly, let $\mathbf{A} \triangleq \{\Phi_{mk}, \phi_{\nu k}\}_{\nu,k}$ be called a set of *analyzing functions.* All functions in $\mathbf{S} \bigcup \mathbf{A}$ are defined from $\mathbf{R}^n$ to $\mathbf{C}$. We will show that if we choose $\mathbf{S}$ and $\mathbf{A}$ appropriately, then any given signal or function, $f : \mathbf{R}^n \to \mathbf{C}$, $f \in L^2$, can be written as follows:

$$f(t) = \sum_{k \in \mathrm{Z}^n} \langle f(t), \Phi_{mk}(t) \rangle \, \Psi_{mk}(t) \; + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathrm{Z}^n} \langle f(t), \phi_{\nu k}(t) \rangle \, \psi_{\nu k}(t). \tag{4}$$

We will call (4) an (inhomogeneous) *Psi–decomposition* of the function f. The homogeneous version will be discussed later. The symbol k in (4) denotes a vector $(k_1, \ldots, k_n) \in \mathrm{Z}^n$. Likewise, the (unstated) argument t of the functions $f, \Phi_{mk}, \Psi_{mk}, \phi_{\nu k}$, and $\psi_{\nu k}$, is a vector $(t_1, \ldots, t_n) \in \mathbf{R}^n$. In (4), f is expressed as a weighted sum of the synthesizing functions in $\mathbf{S}$. The weights in (4) constitute a countable sequence $(\langle f, \Phi_{mk} \rangle, \langle f, \phi_{\nu k} \rangle)$ of complex constants. We will call this sequence the (inhomogeneous) *Phi–transform* of f. The expression (4) is not unlike the Fourier series or the Fourier transform decomposition of f. Since $\hat{f}(\omega) = \langle f, e^{j\omega \cdot t} \rangle$, we can write:

$$f(t) = \int_{\mathbf{R}^n} \left(\langle f(t), e^{j\omega \cdot t} \rangle \, \frac{e^{j\omega \cdot t}}{(2\pi)^n} \right) d\omega. \tag{5}$$

Any decomposition of a signal, like that in (4) or in (5), is interesting because a decomposition defines a transform, and a transform may reveal information that is not obvious in a time–domain representation of the signal. Fourier methods permit but one choice of synthesizing and analyzing functions—the complex exponentials. We will see, however, that in the case of the Psi–decomposition we have considerable lattitude in the choice of our synthesizing and analyzing function sets, $\mathbf{S}$ and $\mathbf{A}$. This freedom of choice can be an asset in application.

The rest of this section consists of two lemmas followed by a theorem. The theorem states our main result that if the sets $\mathbf{S}$ and $\mathbf{A}$ are chosen as prescribed in Lemma 1, then (4) is true for all functions $f \in L^2$.

Lemma 1. *Given an $m \in \mathbf{Z}$; given $\hat{\phi}(\omega)$ such that the properties* P1($\hat{\phi}$), P2($\hat{\phi}$), *and* P3($\hat{\phi}$), *below are true; given $\hat{\Phi}(\omega)$ such that* P1($\hat{\Phi}$), P4($\hat{\Phi}$), *and* P5($\hat{\Phi}$), *are true; $\exists \hat{\psi}(\omega)$ satisfying* P1($\hat{\psi}$) *and* P2($\hat{\psi}$); *and $\exists \hat{\Psi}(\omega)$ satisfying* P1($\hat{\Psi}$) *and* P4($\hat{\Psi}$); *such that $\forall \omega \in \mathbf{R}^n$,*

$$\overline{\hat{\Phi}_m(\omega)}\,\hat{\Psi}_m(\omega) + \sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu(\omega)}\,\hat{\psi}_\nu(\omega) \equiv 1. \tag{6}$$

where, for some constant $c \in \mathbf{R}^+$,

P1($\hat{\phi}$): $\hat{\phi} \in \mathcal{C}^\infty$.

P2($\hat{\phi}$): supp $\hat{\phi}(\omega) \subseteq \{\omega : \pi/4 \leq \|\omega\| \leq \pi\}$.

P3($\hat{\phi}$): $|\hat{\phi}(\omega)| \geq c$, *for* $\omega \in \{\omega : (3\pi/8) - \epsilon \leq \|\omega\| \leq (3\pi/4) + \epsilon\}$; *some* $\epsilon \in \mathbf{R}^+$.

P4($\hat{\Phi}$): supp $\hat{\Phi}(\omega) \subseteq \{\omega : \|\omega\| \leq \pi\}$.

P5($\hat{\Phi}$): $|\hat{\Phi}(\omega)| \geq c$, *for* $\omega \in \{\omega : \|\omega\| \leq (3\pi/4) + \epsilon\}$; *some* $\epsilon \in \mathbf{R}^+$.

Moreover, the functions Ψ and ψ are rapidly-decreasing.

Before we prove Lemma 1, we would like to explain its statement, informally, as follows: this lemma affirms the existence of functions $\hat{\psi}$ and $\hat{\Psi}$ satisfying (6), when we are given $\hat{\phi}$ and $\hat{\Phi}$ that cover the frequency–domain as in Figure 1. The property P2($\hat{\phi}$) is designed to ensure a compact support for $\hat{\phi}(\omega)$; P4($\hat{\Phi}$) to do the same for $\hat{\Phi}(\omega)$. P3($\hat{\phi}$) is designed to ensure that the frequency

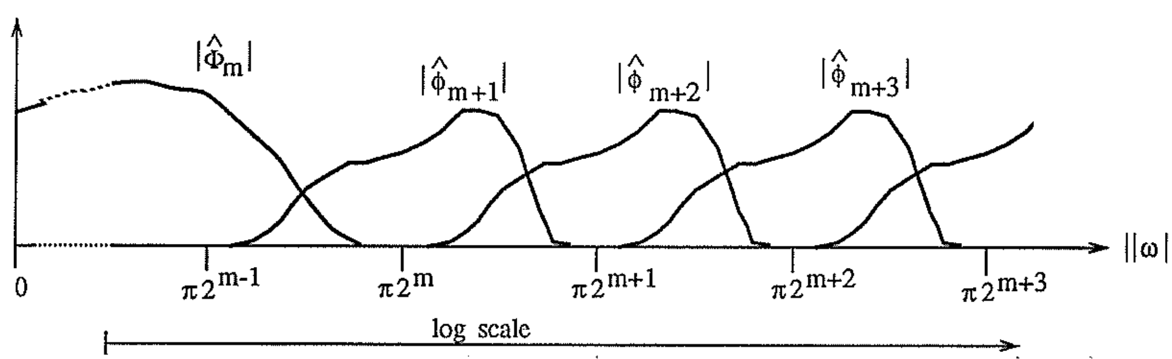

Figure 1: We want the Fourier transforms of the analyzing functions in **A** to cover the frequency space thus.

space is nicely covered—that there is no "bald patch" between $\hat{\phi}_\nu$ and $\hat{\phi}_{\nu+1}$, for any ν. P5($\hat{\Phi}$) ensures that $\hat{\Phi}_m$ properly caps the space left uncovered by all the $\hat{\phi}_\nu$, $\nu \in \{m+1, m+2, \ldots\}$. The functions $\hat{\phi}$ and $\hat{\Phi}$ are not required to be radially–symmetric. Figure 1 should be seen as a representative radial slice across the frequency space. Because $\hat{\phi}$ and $\hat{\Phi}$ are C^∞ and have compact support, ϕ and Φ are C^∞ and rapidly–decreasing [20]. This rapidly–decreasing character of ϕ and Φ, together with the compact support of $\hat{\phi}$ and $\hat{\Phi}$, provides us with the means for determining the time–frequency behaviour of a signal, as we shall see.

Proof of Lemma 1. Let $\hat{\theta}(\omega)$ be a C^∞ function from $\mathbf{R}^n$ to $\mathbf{R}^+$ satisfying P3($\hat{\theta}$); and satisfying *supp* $\hat{\theta}(\omega) \subseteq \{\omega\colon |\hat{\phi}(\omega)| > c/2\} \bigcap \{\omega\colon |\hat{\Phi}(\omega)| > c/2\}$. Because $\hat{\theta}(\omega)$ satisfies P3($\hat{\theta}$), and because its values lie in $\mathbf{R}^+$, we have $\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega) \geq c > 0$, $\forall \omega \neq 0$. Define

$$\hat{\psi}(\omega) \triangleq \begin{cases} \hat{\theta}(\omega) \,/\, (\overline{\hat{\phi}(\omega)} \sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)) & , \omega \in supp\ \hat{\phi}(\omega) \\ 0 & , \text{otherwise.} \end{cases} \tag{7}$$

Because *supp* $\hat{\theta}(\omega) \subseteq \{\omega\colon |\hat{\phi}(\omega)| > c/2\}$, the ratio in (7) is well–defined. Because $\hat{\theta}$ is compactly supported, and because $\hat{\phi}$ and $\hat{\theta}$ are C^∞, $\hat{\psi}$ is C^∞ with compact support. Hence $\hat{\psi}$ is the Fourier transform of some ψ which is C^∞ and rapidly–decreasing. The function $\hat{\psi}(\omega)$ satisfies P2($\hat{\psi}$) since, by the definition in (7), *supp* $\hat{\psi}(\omega) \subseteq$ *supp* $\hat{\theta}(\omega)$. Define

$$\hat{\Psi}_m(\omega) \triangleq \begin{cases} (\sum_{\nu=-\infty}^{m} \hat{\theta}_\nu(\omega)) \,/\, (\overline{\hat{\Phi}_m(\omega)} \sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)) & , \omega \neq 0, \text{and } \omega \in supp\ \hat{\Phi}_m \\ 0 & , \omega \neq 0, \text{and } \omega \notin supp\ \hat{\Phi}_m \\ 1 \,/\, \overline{\hat{\Phi}(0)} & , \omega = 0. \end{cases} \tag{8}$$

Because *supp* $\hat{\theta}(\omega) \subseteq \{\omega : |\hat{\Phi}(\omega)| > c/2\}$, the first ratio in (8) is well–defined. Moreover, $\hat{\Psi}(\omega)$ satisfies P4($\hat{\Psi}$), since *supp* $\hat{\Psi}_m(\omega) \subseteq$ *supp* $\hat{\Phi}_m(\omega)$ by the definition in (8). Like ψ, Ψ is C^∞ and rapidly–decreasing. From (7), we have $\forall \omega \neq 0$,

$$\sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu(\omega)} \hat{\psi}_\nu(\omega) = \sum_{\nu=m+1}^{\infty} \left(\frac{\hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_{j+\nu}(\omega)} \right) = \frac{\sum_{\nu=m+1}^{\infty} \hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)}. \tag{9}$$

From (8), we have $\forall \omega \neq 0$,

$$\overline{\hat{\Phi}_m(\omega)} \hat{\Psi}_m(\omega) = \frac{\sum_{\nu=-\infty}^{m} \hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)}. \tag{10}$$

Keeping in mind the fact that for $\omega = 0$ the left hand sides in the equations (9) and (10) are 0 and 1, respectively; and upon summing these two equations; we find that (6) is true for all ω. □

This proof is constructive. Having decided upon a suitable set of analyzing functions, we can build a set of synthesizing functions by following the recipe in the proof. Alternatively, we may find it convenient to choose the analyzing and synthesizing functions directly so as to satisty (6), without going through the explicit construction suggested in the proof. We will see an example of such a direct choice later in Section 5.

Lemma 2 below establishes a result used in the proof of the main theorem. It uses a technique similar to that used by Shannon [10] in his proof of the sampling theorem.

Lemma 2. *Let* supp $\hat{g}$, supp $\hat{h} \subseteq \{\omega : ||\omega|| \leq \pi 2^{\nu}\}$, $\nu \in \mathbf{Z}$; *and* $\hat{g}, \hat{h} \in L^2$. *Then for* $s, t \in \mathbf{R}^n$, *we can write the convolution of* g *and* h *as:*

$$(g * h)(t) \triangleq \int_{\mathbf{R}^n} g(s)\, h(t-s)\, ds = \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu})\, h(t - k2^{-\nu}). \tag{11}$$

Proof of Lemma 2. By the statement of the lemma, *supp* $\hat{g} \subseteq [-\pi 2^{\nu}, \pi 2^{\nu}]^n$. Let $\hat{g}^c$ be a periodic continuation of $\hat{g}$, so that

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} \hat{g}(\omega - k 2\pi 2^{\nu}). \tag{12}$$

Then $\hat{g}^c(\omega + \Omega) = \hat{g}^c(\omega)$ for $\Omega = \pi 2^{\nu+1} W$; any $W \in \mathbf{Z}^n$. We can now expand $\hat{g}^c(\omega)$ in a Fourier series:

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} a_k \left(\prod_{i=1}^{n} e^{-j2\pi k_i \omega_i / \pi 2^{\nu+1}}\right) = \sum_{k \in \mathbf{Z}^n} a_k e^{-j\omega \cdot k 2^{-\nu}}, \tag{13}$$

where

$$a_k = \pi^{-n} 2^{-n(\nu+1)} \int_{[-\pi 2^{\nu}, \pi 2^{\nu}]^n} \hat{g}^c(\omega)\, e^{j\omega \cdot k 2^{-\nu}}\, d\omega. \tag{14}$$

Since $\hat{g}(\omega) = \hat{g}^c(\omega)$ within the interval of integration in (14), we can replace $\hat{g}^c$ by $\hat{g}$ in (14) above. Further, since *supp* $\hat{g}(\omega) \subseteq [-\pi 2^{\nu}, \pi 2^{\nu}]^n$, we can write (14) as:

$$a_k = \pi^{-n} 2^{-n(\nu+1)} \int_{\mathbf{R}^n} \hat{g}(\omega)\, e^{j\omega \cdot k 2^{-\nu}}\, d\omega = 2^{-n\nu} g(k 2^{-\nu}). \tag{15}$$

Substituting (15) into (13),

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu}) e^{-j\omega \cdot k2^{-\nu}}. \tag{16}$$

Since *supp* $\hat{h}(\omega) \subseteq \{\omega : \|\omega\| \leq \pi 2^\nu\}$,

$$(g * h)(t) = (\hat{g}\hat{h})^\vee(t) = (\hat{g}^c \hat{h})^\vee(t). \tag{17}$$

From (16) and (17) we get the desired result:

$$\begin{aligned}
(g * h)(t) &= [\sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu}) e^{-j\omega \cdot k2^{-\nu}} \hat{h}(\omega)]^\vee(t) && (18)\\
&= \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu}) [\hat{h}(\omega) e^{-j\omega \cdot k2^{-\nu}}]^\vee(t) && (19)\\
&= \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu}) h(t - k2^{-\nu}). && (20)
\end{aligned}$$

□

Theorem 1 (see [7], p. 780). *Given $\hat{\phi}$, $\hat{\Phi}$, $\hat{\psi}$, and $\hat{\Psi}$, satisfying all conditions in the statement of Lemma 1; and given a function $f \in L^2$, $f : \mathbf{R}^n \to \mathbf{C}$; we can write:*

$$f = \sum_{k \in \mathbf{Z}^n} \langle f, \Phi_{mk} \rangle \Psi_{mk} \; + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k} \rangle \psi_{\nu k}. \tag{21}$$

Proof of Theorem. From Lemma 1, $\overline{\hat{\Phi}_m}\, \hat{\Psi}_m + \sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu}\, \hat{\psi}_\nu \equiv 1$. Taking the inverse Fourier transform,

$$\tilde{\Phi}_m * \Psi_m + \sum_{\nu=m+1}^{\infty} \tilde{\phi}_\nu * \psi_\nu = \delta, \tag{22}$$

where $\tilde{g}\,(t) = \overline{g(-t)}$. Convolving both sides with f,

$$f = f * \tilde{\Phi}_m * \Psi_m + \sum_{\nu=m+1}^{\infty} f * \tilde{\phi}_\nu * \psi_\nu. \tag{23}$$

Since *supp* $(f * \tilde{\Phi}_m)\hat{}\; \subseteq$ *supp* $\hat{\Phi}_m$; *supp* $\hat{\Phi}_m$, *supp* $\hat{\Psi}_m \subseteq \{\omega : \|\omega\| \leq \pi 2^m\}$; and $(f * \tilde{\Phi}_m)\hat{}$, $\hat{\Psi}_m \in L^2$; we have (using Lemma 2):

$$(f * \tilde{\Phi}_m) * \Psi_m = \sum_{k \in \mathbf{Z}^n} 2^{-nm} (f * \tilde{\Phi}_m)(k2^{-m})\; \Psi_m(t - k2^{-m}). \tag{24}$$

Simplifying the convolution within the summation above,

$$(f * \widetilde{\Phi}_m)(k2^{-m}) = (f * \widetilde{\Phi}_m)(t)\,|_{t=k2^{-m}} \tag{25}$$

$$= \int_{\mathbf{R}^n} f(t')\,\overline{\Phi_m(t' - k2^{-m})}\,dt' = 2^{nm/2}\langle f, \Phi_{mk}\rangle. \tag{26}$$

Substituting (26) into (24),

$$f * \widetilde{\Phi}_m * \Psi_m = \sum_{k\in\mathbf{Z}^n} 2^{-nm/2}\,\langle f, \Phi_{mk}\rangle\,\Psi_m(t - k2^{-m}) = \sum_{k\in\mathbf{Z}^n} \langle f, \Phi_{mk}\rangle\,\Psi_{mk}. \tag{27}$$

Similarly,

$$f * \widetilde{\phi}_\nu * \psi_\nu = \sum_{k\in\mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle\,\psi_{\nu k}. \tag{28}$$

Substituting (27) and (28) into (23) we get the desired result. □

4. Some Properties of the Psi–decomposition

Theorem 1 says that the set $\mathbf{S} = \{\Psi_{mk}, \psi_{\nu k}\}_{\nu,k}$ is *complete* in L^2, i.e. any function in L^2 can be written as a sum of elements in $\mathbf{S}$. It is easy to see, however, that $\mathbf{S}$ is not a basis for L^2. It follows that every function can be written as the sum of the elements of $\mathbf{S}$ in an infinity of different ways. There is an infinity of the sequences $(\langle f, \Phi_{mk}\rangle, \langle f, \phi_{\nu k}\rangle) \in l^2$ that correspond to every $f \in L^2$. Yet Theorem 1 defines a single–valued function from L^2 to l^2, where the image of f is determined uniquely by the inner–products in (21).

The equality in the Psi–decomposition of (21) is meant in the sense of (3), for $f \in L^2$, where

$$f_N \triangleq \sum_{k \in \mathbf{Z}^n} \langle f, \Phi_{mk}\rangle \Psi_{mk} + \sum_{\nu=m+1}^{N} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle \psi_{\nu k}. \tag{29}$$

In fact, as in the case of the Fourier transform [20], if it is assumed that $\int_{\mathbf{R}^n} |f(x)|\, dx < \infty$ and $\int_{\mathbf{R}^n} |\hat{f}(\omega)|\, d\omega < \infty$, then the representation of f in (21) holds pointwise: $f(x) = lim_{N\to\infty} f_N(x)$, for every $x \in \mathbf{R}^n$. If f is not continuous, the convergence of f_N to f cannot be uniform. In this case the representation (29) will show Gibb's phenomenon.

There is also a *homogeneous* version of the Psi–decomposition. For any $f \in \mathcal{S}'$, we can write

$$f = \sum_{\nu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle\, \psi_{\nu k} \tag{30}$$

with the understanding that the equality is meant "modulo polynomials" (see e.g. [9], appendix B.4). By that we mean that if $P(t)$ is any polynomial in t with complex coefficients, then the decomposition of f and $f + P$ will be identical. This is only to be expected since the support of the Fourier transform of P is $\{\omega = 0\}$. If $f \in L^2$, the homogeneous version still converges in the L^2 sense with partial sums

$$f_N \triangleq \sum_{\nu=-N}^{+N} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle \psi_{\nu k}. \tag{31}$$

While neither of the sets **A** or **S** is orthogonal in our construction of Lemma 1, these sets could be designed to satisfy the following two conditions: $\Phi = \Psi$, and $\phi = \psi$. To do this, let $\hat{\Phi}(\omega)$ satisfy the following properties (see Fig. 2):

Q1: $\hat{\Phi}(\omega) \in \mathbf{R}$, for $\omega \in \mathbf{R}^n$.

Q2: *supp* $\hat{\Phi}(\omega) \subseteq \{\omega\colon \|\omega\| \leq \omega_1\}$; some $\omega_1 \in \mathbf{R}^+$.

Q3: $\hat{\Phi}(\omega) \equiv 1$, for $\omega \in \{\omega\colon \|\omega\| \leq \omega_2\}$; $\omega_1 > \omega_2 \in \mathbf{R}^+$.

Q4: $\|\omega'\| < \|\omega''\| \Rightarrow \hat{\Phi}(\omega'') \leq \hat{\Phi}(\omega')$.

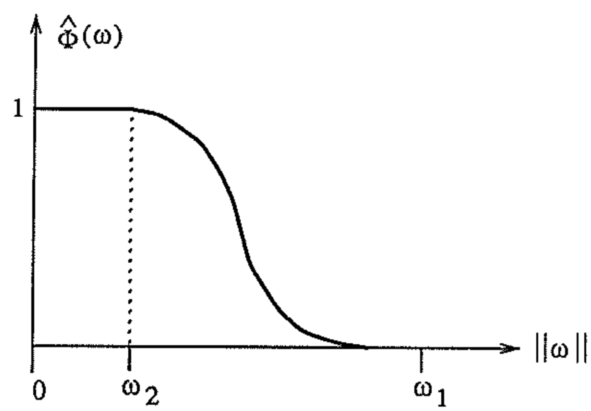

Figure 2: A $\hat{\Phi}(\omega)$ that satisfies properties Q1–Q4.

Define $\hat{\phi}^2(\omega) \triangleq \hat{\Phi}^2(2^{-1}\omega) - \hat{\Phi}^2(\omega)$, for $\omega \in \mathbf{R}^n$. By the monotonicity property Q4, we know that $\hat{\phi}(\omega)$ is real. Then,

$$\sum_{\nu=0}^{N} |\hat{\phi}_\nu(\omega)|^2 = \sum_{\nu=0}^{N} \hat{\phi}_\nu^2(\omega) = \sum_{\nu=0}^{N} \left[\hat{\Phi}^2(2^{-(\nu+1)}\omega) - \hat{\Phi}^2(2^{-\nu}\omega)\right] \tag{32}$$

$$= \hat{\Phi}^2(2^{-(N+1)}\omega) - \hat{\Phi}^2(\omega). \tag{33}$$

Define $K_N(\omega) \triangleq \hat{\Phi}^2(\omega) + \sum_{\nu=0}^{N} |\hat{\phi}_\nu(\omega)|^2$. Then by (33), $K_N(\omega) = \hat{\Phi}^2(2^{-(N+1)}\omega)$. By property Q3, $K_N(\omega) = \hat{\Phi}^2(2^{-(N+1)}\omega) \equiv 1$, for $\omega \in \{\omega\colon \|\omega\| \leq 2^{N+1}\omega_2\}$. Therefore,

$$K_\infty(\omega) = |\hat{\Phi}(\omega)|^2 + \sum_{\nu=0}^{\infty} |\hat{\phi}_\nu(\omega)|^2 \equiv 1, \text{ for } \omega \in \mathbf{R}^n. \tag{34}$$

Equation (34) is just like equation (6) with $\Phi = \Psi$, and $\phi = \psi$. □

Thus the Phi–transform of a function f, obtained with analyzing functions constructed as above, can be computed by repeatedly filtering f with a set of low–pass filters, with results decimated at each stage. This is similar to the method used by Burt and Adelson [21], [22], in their picture–compression work.

We now look at the energy–conservation properties of the Phi–transform.

Theorem 2 (Parseval–like) . *If* A *and* S *are constructed as in the statement of Lemma 1; if, further,* $\Phi = \Psi$ *and* $\phi = \psi$*; and if* $\mathcal{T}f$ *denotes the transform sequence* $(\langle f, \Phi_{mk}\rangle, \langle f, \phi_{\nu k}\rangle)$ *of a function* f*; then* $||\mathcal{T}f||_{l^2} = ||f||_{L^2}$.

Proof of theorem 2. From (21),

$$\begin{aligned} ||f||^2_{L^2} = \langle f, f\rangle &= \langle \left(\sum_{k\in\mathbf{Z}^n} \langle f, \Phi_{mk}\rangle \Psi_{mk} + \sum_{\nu=m+1}^{\infty} \sum_{k\in\mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle \psi_{\nu k} \right), f\rangle && (35) \\ &= \sum_{k\in\mathbf{Z}^n} \langle f, \Phi_{mk}\rangle\langle \Phi_{mk}, f\rangle + \sum_{\nu=m+1}^{\infty} \sum_{k\in\mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle\langle \phi_{\nu k}, f\rangle && (36) \\ &= \sum_{k\in\mathbf{Z}^n} |\langle f, \Phi_{mk}\rangle|^2 + \sum_{\nu=m+1}^{\infty} \sum_{k\in\mathbf{Z}^n} |\langle f, \phi_{\nu k}\rangle|^2 && (37) \\ &= \langle \mathcal{T}f, \mathcal{T}f\rangle = ||\mathcal{T}f||^2_{l^2}. && (38) \end{aligned}$$

□

In general (i.e. for $\Phi \neq \Psi$, $\phi \neq \psi$) we can show that the Phi–transform of a signal is *norm–equivalent* to the signal; i.e. $||\mathcal{T}f||_{l^2} \approx ||f||_{L^2}$. By that we mean that there exist constants C_1 and C_2 in $\mathbf{R}$, such that for all $f \in L^2$,

$$||f||_{L^2} \leq C_1||\mathcal{T}f||_{l^2}, \text{ and } ||\mathcal{T}f||_{l^2} \leq C_2||f||_{L^2}. \tag{39}$$

Norm–equivalence is a property desired of all transforms.

5. Examples of the Phi–transform

Lemma 1 permits considerable flexibility in the choice of the sets **A** and **S** of analyzing and synthesizing functions. This suggests that the Phi–transform is actually a large family of transforms. In this section we give specific examples of the sets **A** and **S**, and of the transformation and reconstruction of signals defined from **R** to **R**.

In our first example we choose the sets **A** and **S** to satisfy (6) a priori; instead of constructing the set **S** according to the prescription in the proof of Lemma 1. If we choose the "window–function" $\hat{\theta}(\omega)$ such that it is a raised cosine pulse whose argument is the logarithm of the frequency variable,

$$\hat{\theta}(\omega) \triangleq \begin{cases} 1/2(1-\cos(\pi \log_2 \|\omega\|)) & , \pi/4 \leq \|\omega\| \leq \pi \\ 0 & , \text{otherwise} \end{cases} \tag{40}$$

then the sum of all the dilations of $\hat{\theta}$ is 1, for all $\omega \neq 0$. Each of the functions $\hat{\phi}$ and $\hat{\psi}$ can now be chosen to be the square–root of $\hat{\theta}$. The DC cap functions $\hat{\Phi}$ and $\hat{\Psi}$ can be chosen to cover the lower frequencies, so as to satisfy (6) near $\omega = 0$. In Figures 3 and 4 we plot these cosine–of–log functions in the frequency and the time domain, respectively. We should point out that the analyzing functions in this first example do not belong to $\mathcal{C}^\infty$, an assumption in the development of Section 3. The square–root of the cosine–of–log function in (40) has discontinuous first and higher–order derivatives at the edges of its support. The only difficulty this poses is that ϕ, Φ, ψ, and Ψ, are not rapidly–decreasing at infinity.

In our second example we look at the problem of deriving a set of analyzing and synthesizing functions which are optimal in some sense. The importance of the Psi–decomposition for time–frequency analysis lies in the fact that both the analyzing and synthesizing functions are simultaneously localized in both the time and the frequency domain. As is well–known, a function cannot be compactly supported in both domains. However, it is possible to pose an optimization problem in which the function is compactly supported in one domain and is "concentrated" in the other. One example of this kind of function is the prolate spheroidal wave function [23] which is strictly bandlimited on the frequency interval $[-B, B]$, and has the maximum fraction of its energy in the time interval $[-T/2, T/2]$.

In this example, we have constructed analyzing functions which are solutions to the eigenvalue problem

$$P_B P_T f = \lambda f. \tag{41}$$

P_B is a projection operator onto the space of bandpass bandlimited functions, and P_T is a projection operator onto the space of time–limited functions, with spectral support given as in Section 3. It can be shown that such a function solves the following problem: find a bandlimited function with given spectral support with the maximum fraction of energy in a given time interval. We omit the details as our only purpose here is to generate an example.

The difficulty with $\hat{\phi}$ and $\hat{\Phi}$ generated in this way is that they are not continuous. We modify these analyzing functions further by convolving them with a compactly supported pulse in the class C^∞, of the form

$$g(\omega) = \begin{cases} ce^{-(\omega+k)^{-2}}e^{-(\omega-k)^{-2}} & , -k \leq \omega \leq k \\ 0 & , \text{otherwise} \end{cases} \tag{42}$$

where $c, k \in \mathbf{R}^+$ are some constants. This "smoothing" operation yields new functions $\hat{\phi}$ and $\hat{\Phi}$ which are C^∞, yet closely approximate the original, optimally–concentrated, functions. The analyzing functions for this second example are plotted in the frequency and the time domains in Figures 5 and 6.

In both the above constructions the analyzing functions are real and symmetric in both frequency and time domains. The localization of the analyzing functions in the time domain is comparable in both of the examples above. In Figures 7–10 we show the analysis and synthesis of two signals, carried out with the cosine–of–log analyzing and synthesizing functions. The time axis k points to the right, while the frequency axis ν points down. In Figures 7 and 9, the number of Phi–transform coefficients at frequency–level ν is half that at level $(\nu + 1)$. The Phi–transform coefficients are all real since f and the elements of $\mathbf{A}$ are. In the reconstruction pictures in Figures 8 and 10 we plot the partial sums (29) for $m = -5$, and $N = -5$(top) to $N = 1$(bottom).

In Figure 10, notice that the high–frequency information required to pass from the $N = -2$ reconstruction to the $N = -1$ reconstruction is contained in a small number of numerically significant coefficients. This is in contrast to the usual Fourier reconstruction methods, where in general twice as many sample values are required to double the frequency range. This indicates that for signals with localized high–frequency components, it is reasonable to expect the Psi–decomposition to yield data compression.

Lastly, in Figure 11, we plot the difference between the Phi–transform coefficients of the two signals at the top. The signals differ at a very few points, and their Phi-transform differs at a few points too. This behaviour of the Phi–transform is in sharp contrast to Fourier analysis, where change at a single point in a signal reverberates across the entire spectrum.

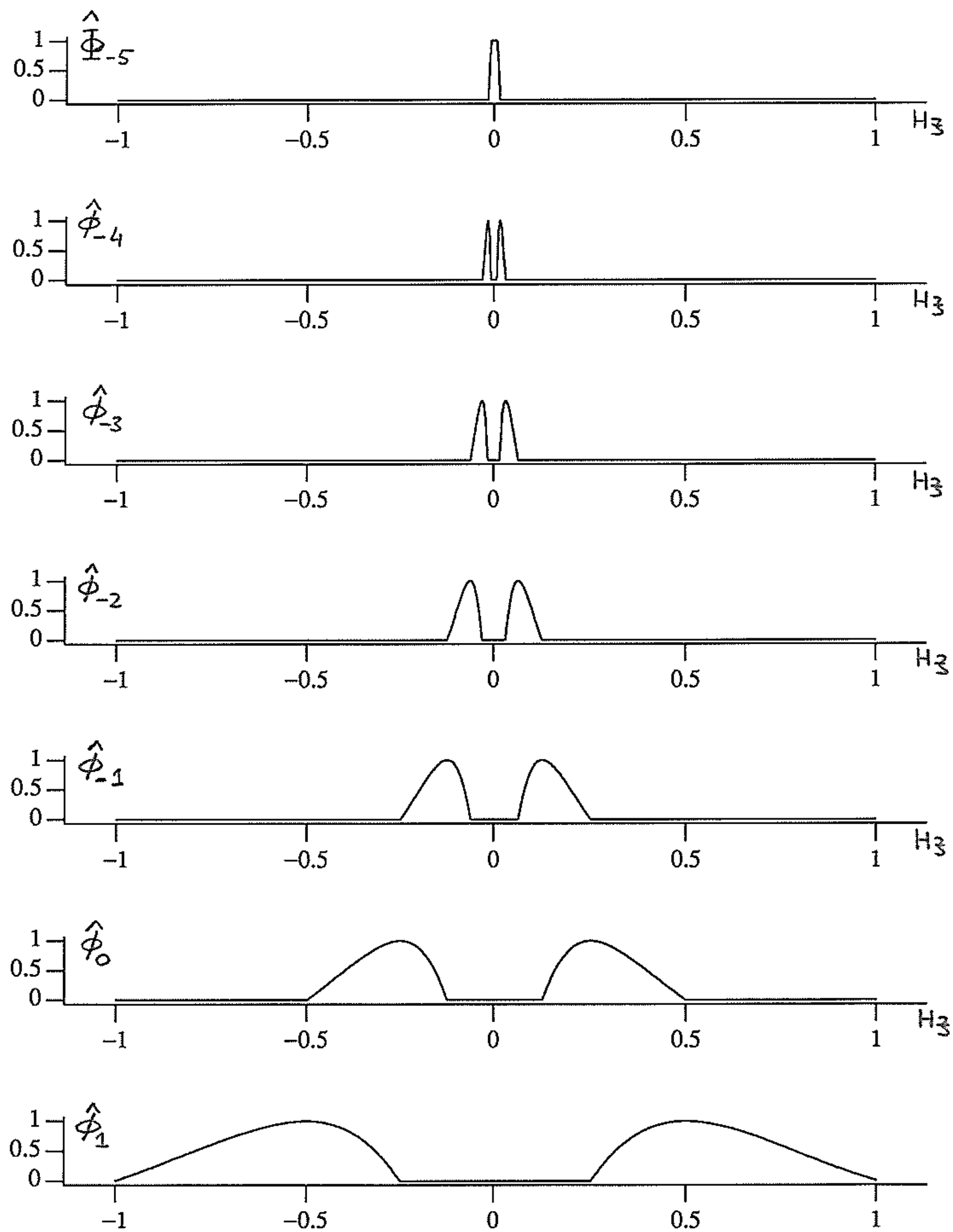

Figure 3: The analyzing/synthesizing functions generated as the cosines-of-logs are plotted here in the frequency domain.

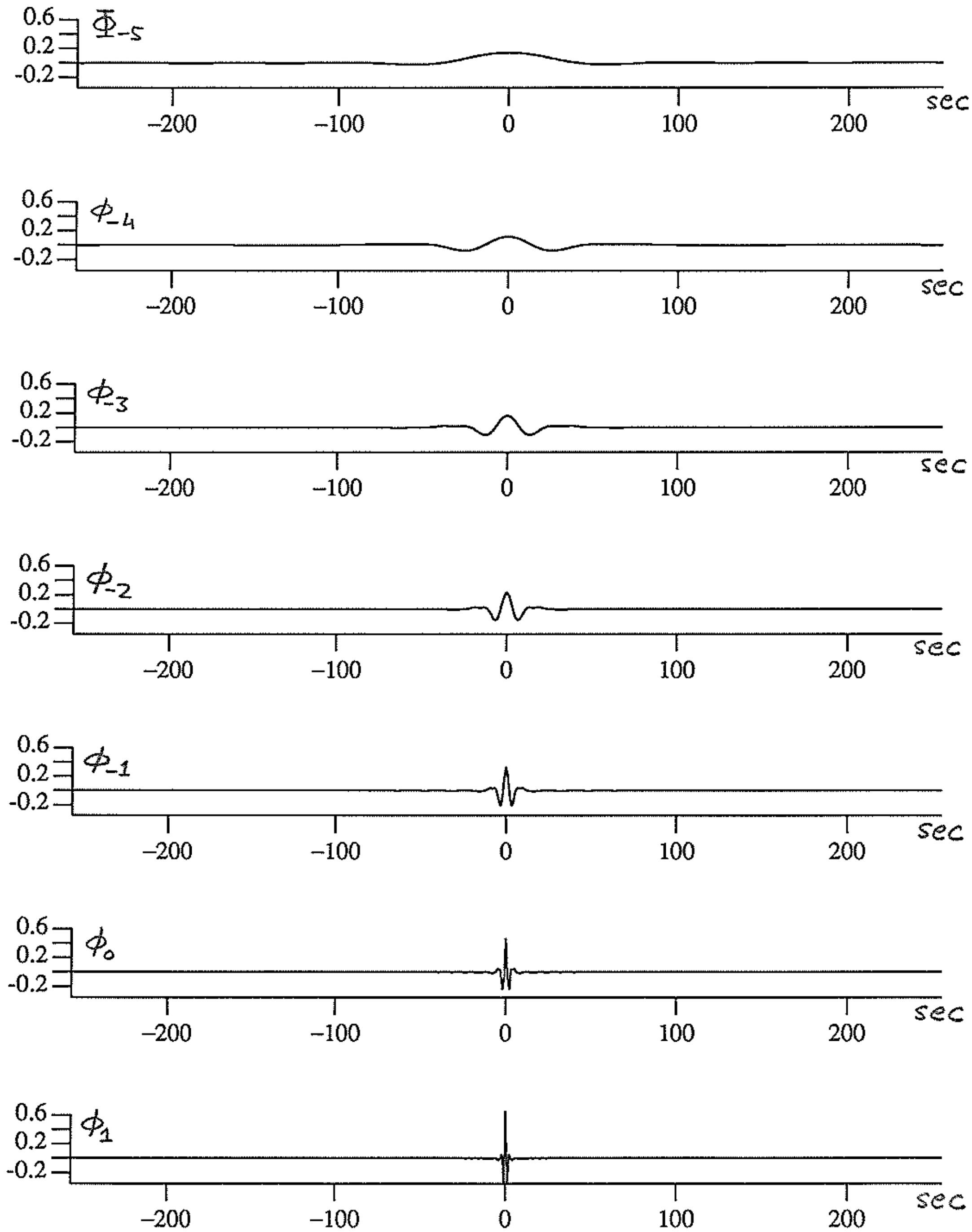

Figure 4: Cosine–of–log analyzing/synthesizing functions in the time domain.

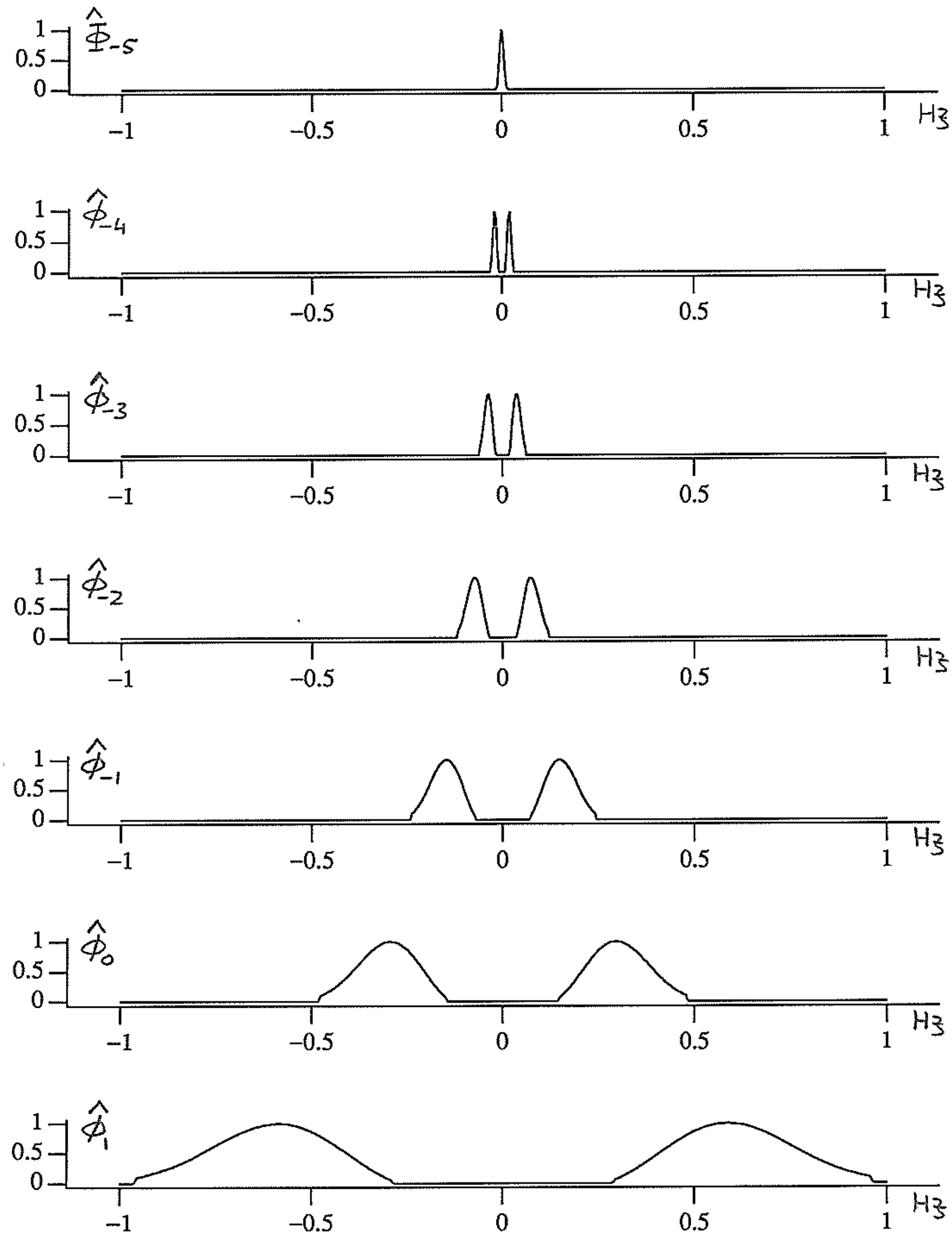

Figure 5: The smoothed–eigenfunction analyzing functions in the frequency domain.

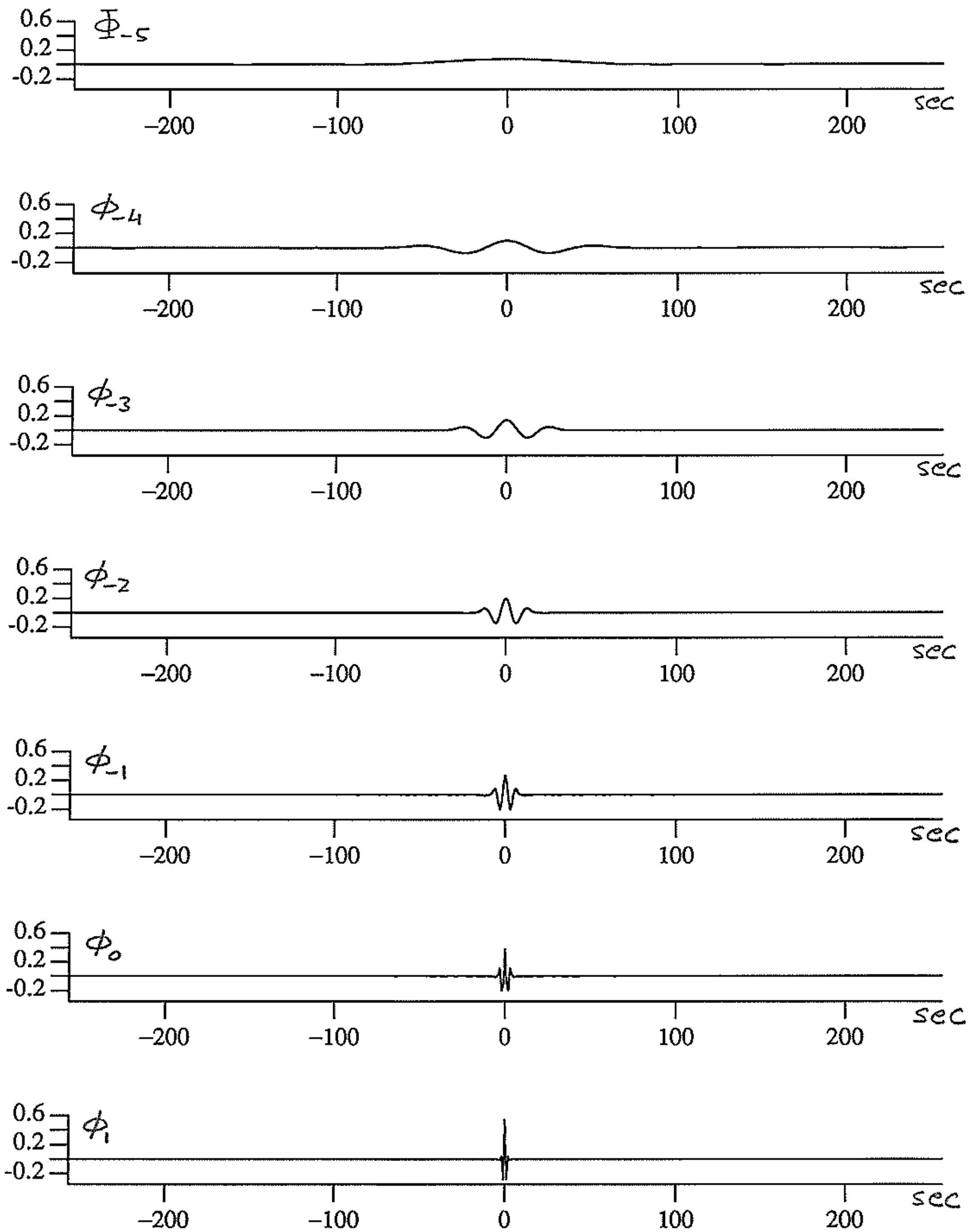

Figure 6: The smoothed-eigenfunction analyzing functions in the time domain.

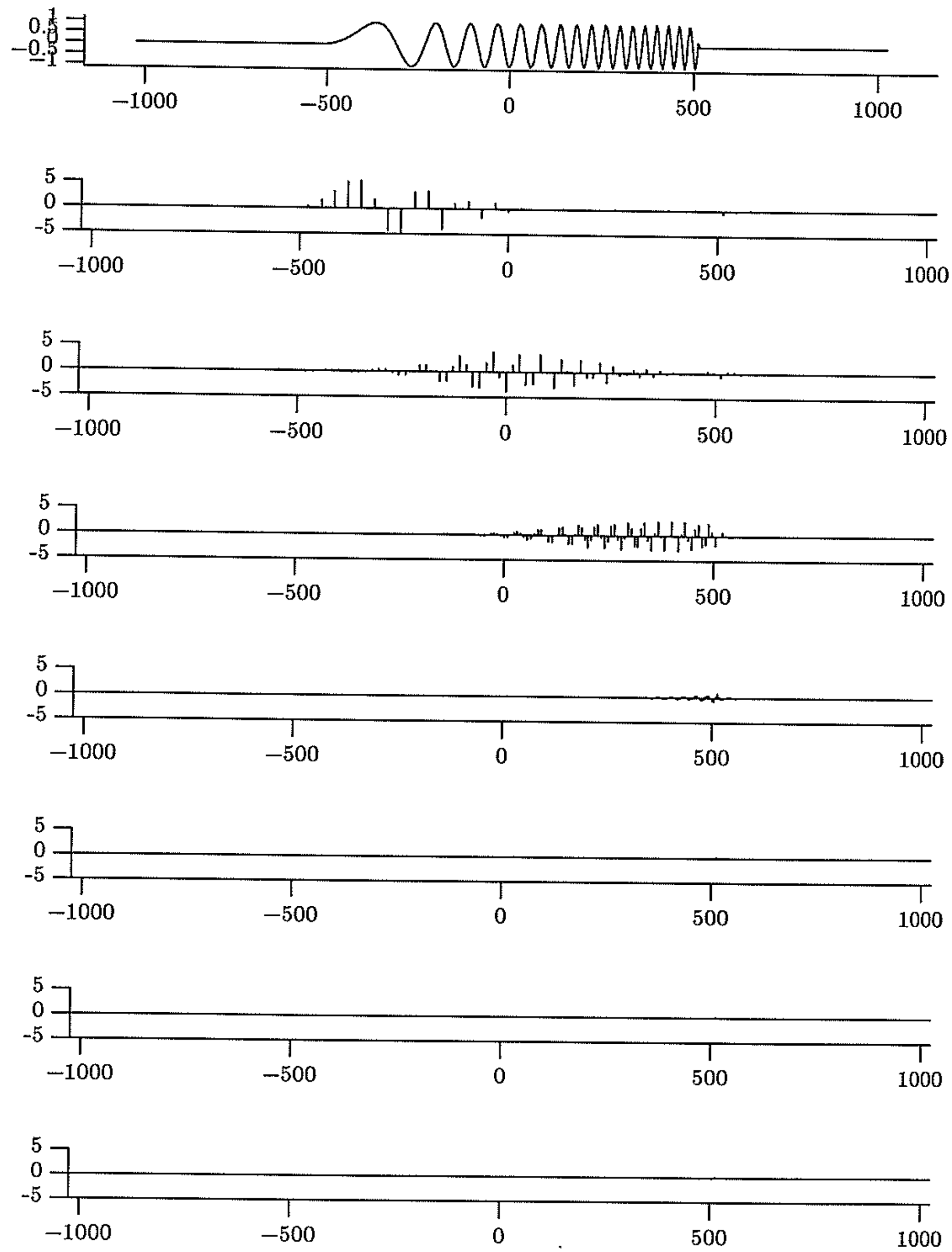

Figure 7: The Phi–transform coefficients of a chirp are computed as the inner–products of the chirp with the analyzing functions.

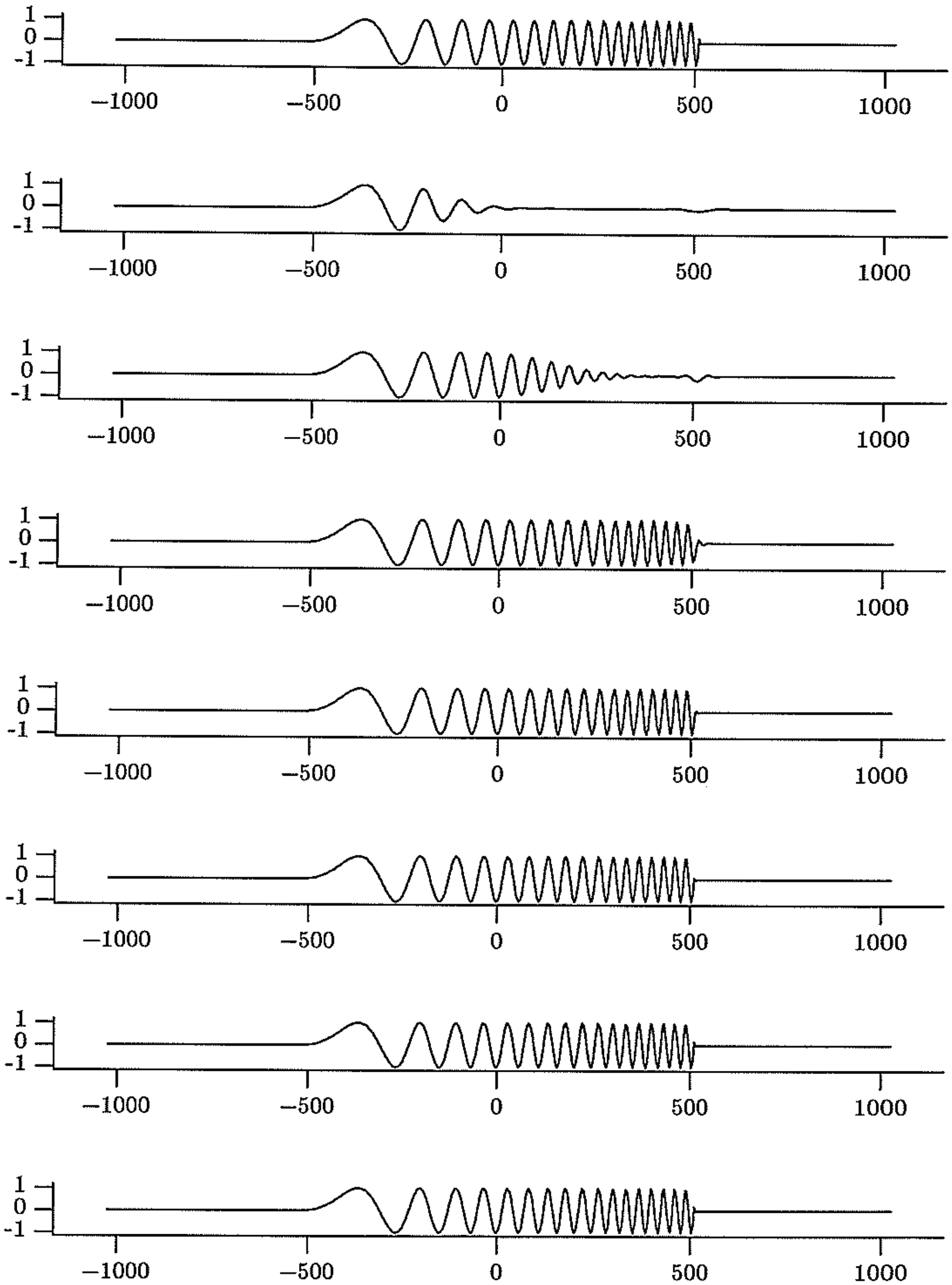

Figure 8: The reconstruction of the chirp from its Phi-transform. The plot on the top is the original signal and the one at the bottom is the reconstructed signal. In between we show the partial reconstructions.

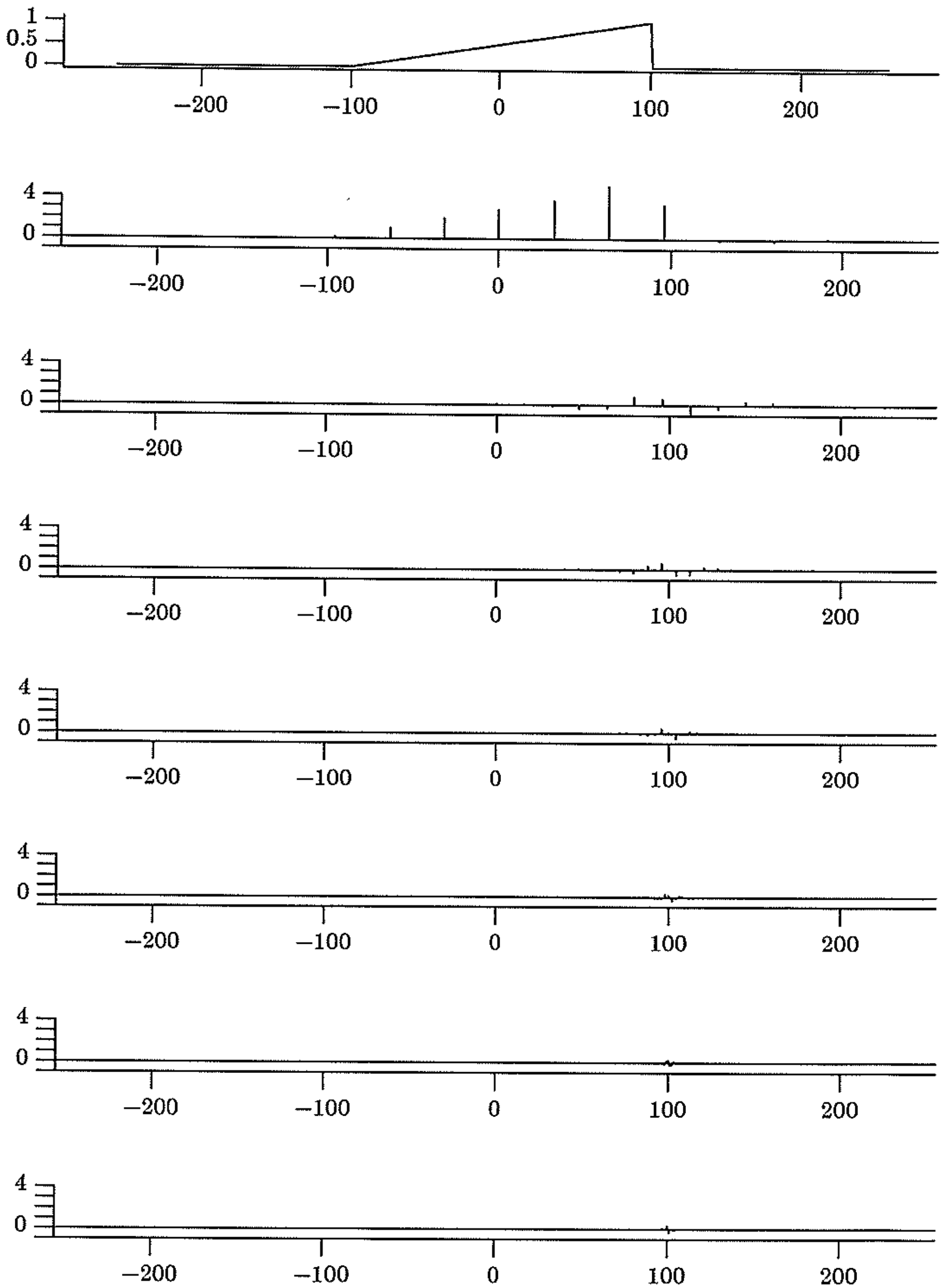

Figure 9: The Phi–transform coefficients of a ramp.

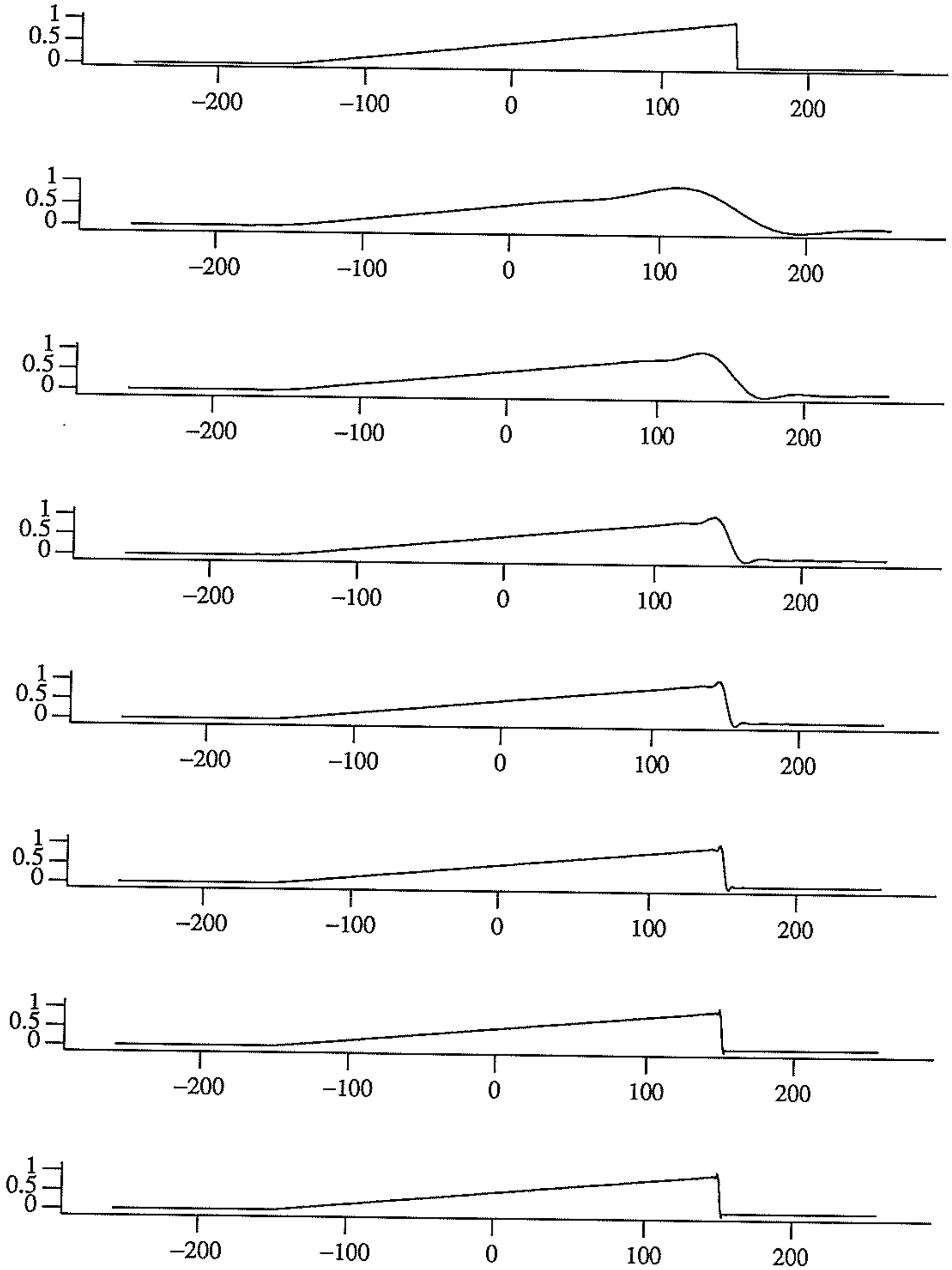

Figure 10: The reconstruction of the ramp from its Phi-transform. Gibb's phenomenon is visible at the point of discontinuity.

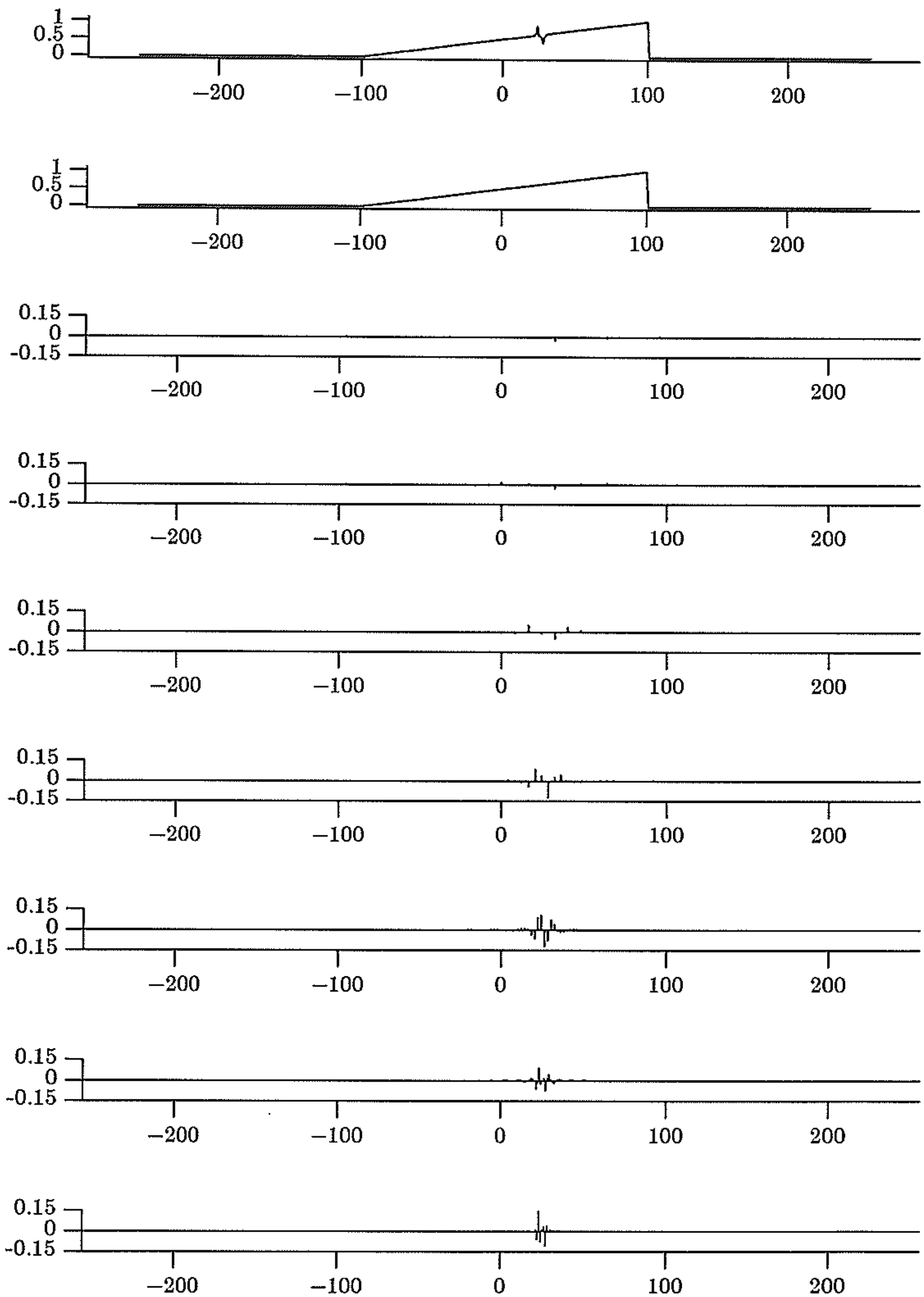

Figure 11: The difference of the Phi–transform coefficients of the two signals (at the top) is plotted. A local difference in the two signals is seen to produce local differences in the Phi–transform coefficients.

6. Conclusions

We have seen that the Phi–transform provides us with a simple and versatile tool for time–frequency analysis, and overcomes the limitations of the classical methods of Gabor and Wigner. The Phi–transform is also very flexible. It gives us considerable control over the design of the analyzing and synthesizing functions, and this control can be exploited in a practical way.

There are more general decompositions [9], similar to the Psi–decomposition, that are not treated here. Methods completely different from those used in Lemma 1 could be used to generate the sets **A** and **S** of analyzing and synthesizing functions [9]. We need not identify the supports of the functions in **A** and **S**. We might require that the elements of **A**, or **S**, or both, be compactly–supported in the time–domain and not in the frequency–domain. Or we may forego compact support in both the time and the frequency domains, requiring only that our functions be small and rapidly decaying outside appropriate compact intervals.

We emphasize that the transform considered in this paper is applicable to functions defined continuously in time, $f \in L^2[\mathbf{R}^n]$. The examples given in the last sections were generated by a discretization of the continuous world. However, the continuous–time Phi–transform is not a computational device; it was developed in order to characterize distribution spaces. One goal of our research is the use of these time–frequency techniques for computational problems such as coding and compression, and we are working on a discrete formalism of the Phi–transform that would work with signals $f \in l^2[\mathbf{Z}^n]$, or with $f \in l^2[\mathbf{Z}^n_m]$.

Our immediate interest in the Phi–transform arises out of work in the compression of moving pictures. In the commercial world of video–conferencing systems, pictures are currently segmented into blocks, the Discrete Cosine Transform (DCT) of each block is computed, and the DCT coefficients are transmitted instead of the space–domain information. The advantage of this scheme is that the DCT coefficients can be quantized differentially in order to exploit the differential response of the eye to different spatial frequencies. This differential quantization—coarser at the higher frequencies—yields compression. This DCT–scheme is really a kind of space–frequency transformation, where space–discrimination is obtained through the artifice of blocking; but this blocking produces an unseemly "blocking artifact" at the receiving end. We expect the Phi–transform to do better because it uses a gentler method to attain spatial discrimination. The Phi–transform can take advantage of differential quantization without generating the "blocking artifact". In the matter of "motion compensation" too we expect the Phi–transform to come out ahead. For small interframe–image–differences the image could be updated through the transmission of a few Phi–transform coefficients; and we may not have to resort to the "motion compensation" schemes used with the DCT.

Other areas in which the Phi and the Wavelet transform are being applied are: the compression of static pictures; the recognition of handwritten characters; statistical models for non–stationary signals in radar and underwater acoustics; the computer–aided design of surfaces; and the solution of partial differential equations. We also contemplate applications to automatic music transcription, and to the design of active noise–cancellation systems.

7. Acknowledgements

We would like to acknowledge many helpful discussions with Professors Jerome R. Cox, Jr., and Steven G. Krantz.

8. Appendix

Fourier series

Let $f(x+X_i) = f(x)$; $\forall x \in \mathbf{R}^n$; and some $X_i \in \mathbf{R}^+, i \in [1,\ldots,n]$. Let $c_i \in \mathbf{R}, i \in [1,\ldots,n]$ be some set of constants. Let $Y = \prod_{i=1}^{n}[c_i, c_i + X_i]$. Then

$$f(x) = \sum_{k \in \mathbf{Z}^n} a_k (\prod_{i=1}^{n} e^{-j2\pi k_i x_i / X_i})$$

where,

$$a_k = (\prod_{i=1}^{n} X_i)^{-n} \int_Y f(x) [\prod_{i=1}^{n} e^{j2\pi k_i x_i / X_i}]\, dx$$

Fourier transform

$$\hat{f}(\omega) = \int_{\mathbf{R}^n} f(t) e^{j\omega \cdot t}\, dt;\ \omega, t \in \mathbf{R}^n$$

$$f(t) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{f}(\omega) e^{j\omega \cdot t}\, d\omega$$

scaling theorems

$$1. (f(at))\hat{}(\omega) = a^{-n}\, \hat{f}(\omega/a), a \in \mathbf{R}$$

$$2. (\hat{f}(b\omega))^{\vee}(t) = b^{-n}\, f(t/b), b \in \mathbf{R}$$

convolution theorems

$$1. (f \otimes g)\hat{}(\omega) = \hat{f}(\omega)\, \hat{g}(\omega)$$

$$2. (\hat{f} \otimes \hat{g})^{\vee}(t) = (2\pi)^n f(t)\, g(t)$$

shift theorems

$$1. (f(t+t_0))\hat{}(\omega) = e^{j\omega \cdot t_0} \hat{f}(\omega)$$

$$2. (\hat{f}(\omega + \omega_0))^{\vee}(t) = e^{-j\omega_0 \cdot t} f(t)$$

conjugation theorems

$$1. (\overline{f(t)})\hat{}(\omega) = \overline{\hat{f}(-\omega)} = \tilde{\hat{f}}\ (\omega)$$

$$2. (\overline{\hat{f}(\omega)})^{\vee}(t) = \overline{f(-t)} = \tilde{f}\ (t)$$

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